

Nonconformal Scalar Field in a Homogeneous Isotropic Space and the Method of Hamiltonian Diagonalization

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Abstract. The diagonalization of the metrical Hamiltonian of a scalar field with an arbitrary coupling with a curvature in N -dimensional homogeneous isotropic space is performed. The energy spectrum of the corresponding quasiparticles is obtained. The energies of the quasiparticles corresponding to the diagonal form of the canonical Hamiltonian are calculated. The modified energy-momentum tensor with the following properties is constructed. It coincides with the metrical energy-momentum tensor for conformal scalar field. Under its diagonalization the energies of relevant particles of a nonconformal field coincide to the oscillator frequencies and the density of such particles created in a nonstationary metric is finite. It is shown that the Hamiltonian calculated with the modified energy-momentum tensor can be constructed as a canonical Hamiltonian under the special choice of variables.

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1. Introduction

Quantum field theory in curved space-time is a deeply worked out part of theoretical physics now (see monographs [1, 2]). It is of large interest due to its important applications to cosmology and astrophysics. At the same time in this theory there are several problems that haven't been finally solved till now. One of them is definition of the notion of elementary particle in curved space-time. It is caused by the absence of the group of symmetries like Poincare group in Minkowsky space. The definition of vacuum state and particles states for nonconformal scalar field is being discussed intensely at

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present even in case of homogeneous isotropic space [3, 4, 5]. As a consequence of various definitions of vacuum states we have variety of calculated quantum characteristics of nonconformal scalar field in curved space.

In works [6, 7] it was shown that in the case of arbitrary coupling of scalar field with curvature additional nonconformal contributions are dominant in vacuum averages of energy-momentum tensor. It should be also mentioned that the investigation of nonconformal scalar field is not of independent interest only. This investigation is caused by impossibility of preservation of conformal invariance of effective action and usual action in the case of interacting quantized field [2].

In the definition of vacuum and in the formulation of the problem of particles creation in curved space-time two following approaches are being widely used: the Hamiltonian diagonalization procedure [1] offered in [8, 9] and the so-called "adiabatic" procedure [2] offered in [10]. Supposing that a quantum of energy corresponds to a particle then observation of particles at some moment (according to quantum mechanics) means to find the Hamiltonian eigenstate. It is taken into account automatically in the diagonalization approach. Making use of Hamiltonian built according to the metrical energy-momentum tensor, having been successful for the conformal case (see [1]). But for nonconformal scalar field in diagonalization approach it led to difficulties, connected with infinite of quasiparticles creation density [11]. At the same time essential problems and ambiguities take place in adiabatic approach also [5].

In this paper the procedure of diagonalization for scalar field with arbitrary coupling with curvature in N -dimensional homogeneous isotropic space is investigated. Carrying out calculations in N -dimensional space-time while finding vacuum state and defining particles states is necessary in particular when the dimensional regularization is used in further. In section 2 necessary information is given and nonconformal scalar field quantizing in N -dimensional homogeneous isotropic space is performed. In section 3 the metrical Hamiltonian diagonalization of nonconformal scalar field is carried out. The energies of corresponding quasiparticles are calculated and conditions connected with demand of the metrical Hamiltonian diagonality are investigated. In section 4 the diagonalization approach is used for the canonical Hamiltonian, the modified energy-momentum tensor is built so that the quasiparticles from diagonalization of corresponding Hamiltonian have energies coinciding with the oscillator frequency of the wave equation. It is

proved that the density of particles being created in nonstationary metric is finite. In section 5 it is shown that the Hamiltonian obtained from the modified energy-momentum tensor can be built as canonical one under a certain choice of canonical variables and the results of performed investigation are summarized.

The system of units in which $\hbar = c = 1$ is using in the paper.

2. Quantizing of Scalar Field in Homogeneous Isotropic Space

We will consider the complex scalar field $\phi(x)$ of mass m obeying the equation

$$(\nabla_i \nabla^i + \xi R + m^2) \phi(x) = 0, \quad (1)$$

where ∇_i is covariant derivative, R is the scalar curvature, $x = (t, \mathbf{x})$, ξ is coupling constant. The value $\xi = \xi_c = (N - 2)/[4(N - 1)]$ correspond to conformal coupling in space-time of dimension N ($\xi_c = 1/6$ if $N = 4$). The equation (1) is conformally invariant if $\xi = \xi_c$ and $m = 0$. The value $\xi = 0$ reduces to the case of minimal coupling.

The metric of N -dimensional homogeneous isotropic space-time is

$$ds^2 = g_{ik} dx^i dx^k = dt^2 - a^2(t) dl^2 = a^2(\eta) (d\eta^2 - dl^2), \quad (2)$$

where $dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta$ is the metric of $N - 1$ dimensional space with the constant curvature $K = 0, \pm 1$.

The equation (1) can be received by varying the action with Lagrangian density

$$L(x) = \sqrt{|g|} [g^{ik} \partial_i \phi^* \partial_k \phi - (m^2 + \xi R) \phi^* \phi], \quad (3)$$

where $g = \det(g_{ik})$.

Canonical energy-momentum tensor of scalar field is

$$T_{ik}^{can} = \partial_i \phi^* \partial_k \phi + \partial_k \phi^* \partial_i \phi - g_{ik} |g|^{-1/2} L(x). \quad (4)$$

The metrical energy-momentum tensor which can be got varying the action in g_{ik} have a form [12]:

$$T_{ik} = T_{ik}^{can} - 2\xi [R_{ik} + \nabla_i \nabla_k - g_{ik} \nabla_j \nabla^j] \phi^* \phi, \quad (5)$$

where R_{ik} is Ricci tensor.

In the metric (2) the equation (1) is

$$\phi'' + (N - 2) \left(\frac{a'}{a} \right) \phi' - \Delta_{N-1} \phi + (m^2 + \xi R) a^2 \phi = 0, \quad (6)$$

where Δ_{N-1} is the Laplace-Beltrami operator in $N - 1$ dimension space, the prime denotes the derivative with conformal time η .

For the function $\tilde{\phi} = a^{(N-2)/2} \phi$ the equation (6) has the form without the first derivative with time

$$\tilde{\phi}'' - \Delta_{N-1} \tilde{\phi} + (m^2 a^2 - \Delta \xi a^2 R + ((N - 2)/2)^2 K) \tilde{\phi} = 0, \quad (7)$$

where $\Delta \xi = \xi_c - \xi$. The variables in the equations (6), (7) can be separated. Namely, for $\tilde{\phi} = g_\lambda(\eta) \Phi_J(\mathbf{x})$ we have

$$g_\lambda''(\eta) + \Omega^2(\eta) g_\lambda(\eta) = 0, \quad (8)$$

$$\Delta_{N-1} \Phi_J = -(\lambda^2 - ((N - 2)/2)^2 K) \Phi_J, \quad (9)$$

where $\Omega(\eta)$ is the oscillator frequency

$$\Omega^2(\eta) = m^2 a^2 + \lambda^2 - \Delta \xi a^2 R, \quad (10)$$

J is a set of indexes (quantum numbers), numbering the eigenfunctions of the Laplace-Beltrami operator. It should be noted that the eigenvalue of the operator $-\Delta_{N-1}$ are not negative and we have the inequality $\lambda^2 - ((N - 2)/2)^2 K \geq 0$.

For quantization we are decomposing the field $\tilde{\phi}(x)$ by the complete set of the equation (7) solutions

$$\tilde{\phi}(x) = \int d\mu(J) \left[\tilde{\phi}_{\bar{J}}^{(-)} a_J^{(-)} + \tilde{\phi}_J^{(+)} a_J^{(+)} \right], \quad (11)$$

where $d\mu(J)$ is the measure in the space of the Laplace-Beltrami Δ_{N-1} eigenvalues

$$\tilde{\phi}_J^{(+)}(x) = \frac{1}{\sqrt{2}} g_\lambda(\eta) \Phi_J^*(\mathbf{x}), \quad \tilde{\phi}_{\bar{J}}^{(-)}(x) = \frac{1}{\sqrt{2}} g_\lambda^*(\eta) \Phi_{\bar{J}}(\mathbf{x}), \quad (12)$$

$\Phi_J(\mathbf{x})$ is orthonormal eigenfunctions of Δ_{N-1} operator, \bar{J} is a set of quantum numbers of the function complex conjugated to the function Φ_J . In $N - 1$ dimensional spherical coordinates for $J = \{\lambda, l, \dots, m\}$ we have $\bar{J} = \{\lambda, l, \dots, -m\}$.

Substituting the decomposition (11) in the equation for conservative charge

$$Q = i \int_{\Sigma} (\tilde{\phi}^* \partial_0 \tilde{\phi} - (\partial_0 \tilde{\phi}^*) \tilde{\phi}) \sqrt{\gamma} d^{N-1}x, \quad (13)$$

where $\gamma = \det(\gamma_{\alpha\beta})$, Σ is a space-like hypersurface $\eta = \text{const}$, under the condition of normalization

$$g_{\lambda} g_{\lambda}^{*'} - g'_{\lambda} g_{\lambda}^* = -2i, \quad (14)$$

we have

$$Q = \int d\mu(J) \left(\tilde{a}_J^{*(+)} a_J^{(-)} - \tilde{a}_{\bar{J}}^{*(-)} a_{\bar{J}}^{(+)} \right). \quad (15)$$

The metrical Hamiltonian can be expressed through the metrical energy-momentum tensor (5) by the equation [1]:

$$\begin{aligned} H(\eta) &= \int_{\Sigma} \zeta^i T_{ik}(x) d\sigma^k = \int_{\eta=\text{const}} \zeta^0 T_{00}(x) g^{00} \sqrt{|g|} d^{N-1}x = \\ &= a^{N-2}(\eta) \int_{\eta=\text{const}} T_{00}(x) \sqrt{\gamma} d^{N-1}x, \end{aligned} \quad (16)$$

where $(\zeta^i) = (1, 0, \dots, 0)$ is the time-like conformal Killing vector.

The quantization is realized by the commutation relations

$$[a_J^{(-)}, \tilde{a}_{J'}^{(+)}] = [\tilde{a}_J^{*(-)}, a_{J'}^{(+)}] = \delta_{JJ'}, \quad [a_J^{(\pm)}, a_{J'}^{(\pm)}] = [\tilde{a}_J^{*(\pm)}, \tilde{a}_{J'}^{*(\pm)}] = 0. \quad (17)$$

The Hamiltonian (16) can be written through $a_J^{(\pm)}$, $\tilde{a}_J^{*(\pm)}$ operators in such form

$$\begin{aligned} H(\eta) &= \int d\mu(J) \left\{ E_J(\eta) \left(\tilde{a}_J^{*(+)} a_J^{(-)} + \tilde{a}_{\bar{J}}^{*(-)} a_{\bar{J}}^{(+)} \right) + \right. \\ &\quad \left. + F_J(\eta) \tilde{a}_J^{*(+)} a_{\bar{J}}^{(+)} + F_J^*(\eta) \tilde{a}_{\bar{J}}^{*(-)} a_J^{(-)} \right\}, \end{aligned} \quad (18)$$

where

$$E_J(\eta) = \frac{1}{2} \left\{ |g'_{\lambda}|^2 + D_{\lambda}(\eta) |g_{\lambda}|^2 - Q(\eta) (|g_{\lambda}|^2)' \right\}, \quad (19)$$

$$F_J(\eta) = \frac{(-1)^m}{2} \left\{ g_{\lambda}'^2 + D_{\lambda}(\eta) g_{\lambda}^2 - Q(\eta) (g_{\lambda}^2)' \right\}, \quad (20)$$

$$D_{\lambda}(\eta) = m^2 a^2 + \lambda^2 + \Delta \xi (N-1)(N-2)(c^2 - K), \quad (21)$$

$$Q(\eta) = \Delta \xi 2(N-1)c, \quad (22)$$

$$c = a'(\eta)/a(\eta).$$

3. The Diagonalization of the Metrical Hamiltonian

The metrical Hamiltonian (18) will be diagonal at time moment η_0 in operators $\hat{a}_J^{*(\pm)}$, $\hat{a}_J^{(\pm)}$, which in this case are the creation and annihilation operators of particles and antiparticles under $F_J(\eta_0) = 0$. Using (20) – (22) we can check that this condition is consistent with normalization (14) only under $p_\lambda^2(\eta_0) > 0$, where

$$p_\lambda(\eta) = \sqrt{m^2 a^2(\eta) + \lambda^2 + 4\Delta\xi (N-1)^2 (\xi c^2 - \xi_c K)}. \quad (23)$$

Requirement of the diagonal form of Hamiltonian at time moment η_0 , i.e. $F_J(\eta_0) = 0$, with the condition of normalization result in the following initial conditions for the function $g_\lambda(\eta_0)$

$$g'_\lambda(\eta_0) = (2\Delta\xi(N-1)c + ip_\lambda(\eta_0))g_\lambda(\eta_0), \quad |g_\lambda(\eta_0)| = 1/\sqrt{p_\lambda(\eta_0)}. \quad (24)$$

The state of vacuum $|0\rangle$ corresponding to (24) is defined in a standard way

$$a_J^{(-)}|0\rangle = \hat{a}_J^{(-)}|0\rangle = 0. \quad (25)$$

For arbitrary time moment η we will diagonalize the Hamiltonian in terms of $\hat{b}_J^{*(\pm)}$, $\hat{b}_J^{(\pm)}$ operators which connected with $\hat{a}_J^{*(\pm)}$, $\hat{a}_J^{(\pm)}$ by the time-dependent Bogolyubov transformations:

$$\begin{cases} \hat{a}_J^{(-)} = \alpha_J^*(\eta) \hat{b}_J^{(-)}(\eta) - (-1)^m \beta_J(\eta) \hat{b}_J^{(+)}(\eta), \\ \hat{a}_J^{*(-)} = \alpha_J^*(\eta) \hat{b}_J^{*(-)}(\eta) - (-1)^m \beta_J(\eta) \hat{b}_J^{*(+)}(\eta), \end{cases} \quad (26)$$

where $\alpha_J(\eta)$, $\beta_J(\eta)$ is the functions satisfy the initial conditions $|\alpha_J(\eta_0)| = 1$, $\beta_J(\eta_0) = 0$ and the identity

$$|\alpha_J(\eta)|^2 - |\beta_J(\eta)|^2 = 1. \quad (27)$$

(In homogeneous and isotropic space $\alpha_J = \alpha_\lambda$, $\beta_J = \beta_\lambda$ [1]).

Let's substitute decomposition (26) in (18). If we demand coefficients before nondiagonal terms $\hat{b}_J^{*(\pm)}\hat{b}_J^{(\pm)}$ equal to 0 we'll come to equation

$$2(-1)^{m+1}\alpha_J\beta_J E_J + F_J\alpha_J^2 + F_J^*\beta_J^2 = 0. \quad (28)$$

It can be shown that the condition (28) is consistent with normalization (14) only if $p_\lambda^2(\eta) > 0$. In that case

$$|\beta_J|^2 = E_J/(2p_\lambda) - 1/2 = |F_J|^2/(2p_\lambda(E_J + p_\lambda)). \quad (29)$$

Concluding (29) we took into account the equation that could be easily checked:

$$E_J^2 - |F_J|^2 = p_\lambda^2(\eta) \cdot [-(g_\lambda g_\lambda^{*'} - g_\lambda' g_\lambda^*)^2 / 4] . \quad (30)$$

(The multiplier in square brackets equals to 1 under the normalization condition (14)).

In the case of (28) and $p_\lambda^2(\eta) > 0$ the metrical Hamiltonian takes the form

$$H(\eta) = \int d\mu(J) p_\lambda(\eta) \left(b_J^{*(+)} b_J^{(-)} + b_J^{*(-)} b_J^{(+)} \right) . \quad (31)$$

So $p_\lambda(\eta)$ has the meaning of energy of quasiparticles corresponding to the diagonal form of the metrical Hamiltonian. For the 4 dimensional space-time the equation (23) corresponds to energy values obtained in [3] and [13].

Quasiparticles' energy $p_\lambda(\eta)$ differ from oscillator frequency $\Omega(\eta)$ of the wave equation for nonconformal field. It leads to a series of difficulties. Thus the conditions $p_\lambda^2(\eta) > 0$ and $\Omega^2(\eta) > 0$ may be in contradiction for nonconformal field in some case. For example, in the case of quasi-euclidean space ($K = 0$) and zero-mass field the condition $p_\lambda^2(\eta) > 0$ (with arbitrary λ) reduces to $\xi \in [0, \xi_c]$. But if $\xi < \xi_c$, $m = 0$ and $R > 0$ for low λ we have $\Omega^2(\eta) < 0$.

It should be noted that for $p_\lambda^2(\eta) < 0$ the condition of diagonalization reduces to vanishing of norm, energy and charge of the state with $\phi(x) \neq 0$, that hasn't any physical foundation.

The vacuum state defined by equations

$$b_J^{(-)} |0_\eta\rangle = b_J^{*(-)} |0_\eta\rangle = 0 , \quad (32)$$

depends on time in nonstationary metric. Under initial conditions (24) $b_J^{(\pm)}(\eta_0) = a_J^{(\pm)}$ and $|0_{\eta_0}\rangle = |0\rangle$. In the Heisenberg representation the state $|0\rangle$ is vacuum at time moment η_0 but it isn't vacuum if $\eta \neq \eta_0$. It has $|\beta_J(\eta)|^2$ pairs of quasiparticles corresponding to operators $b_J^{*(\pm)}, b_J^{(\pm)}$ [1] in each mode. The number of the created pairs of quasiparticles in the unit of space volume (for $N = 4$) is

$$n(\eta) = \frac{1}{2\pi^2 a^3(\eta)} \int d\mu(J) |\beta_\lambda(\eta)|^2 . \quad (33)$$

For the metrical Hamiltonian of nonconformal field the density of quasiparticles created in nonstationary metric (33) is infinite [11]. It follows from asymptotic behaviour in $\lambda \rightarrow \infty$ of equation's (8) solutions and equation (29).

4. Diagonalization of the Canonical Hamiltonian and the Modified Energy-Momentum Tensor

The Hamiltonian corresponding to canonical energy-momentum tensor (4) can be written in form (18) with change of (21) and (22) by

$$D_\lambda(\eta) = m^2 a^2 + \lambda^2 + (N-1)(N-2) \left((\xi + \xi_c) c^2 - \Delta \xi K \right) + 2 \xi (N-1) c', \quad (34)$$

$$Q(\eta) = c(N-2)/2. \quad (35)$$

That's why the procedure of the canonical Hamiltonian diagonalization can be carried out as in the case of the metrical Hamiltonian (see section 3) with new functions $D_\lambda(\eta)$ and $Q(\eta)$. One can be convinced that the requirement of the Hamiltonian diagonality is consistent now to normalization (14) only in the case of $p_{can,\lambda}^2(\eta) > 0$, where

$$p_{can,\lambda}(\eta) = \sqrt{(m^2 + \xi R) a^2(\eta) + \lambda^2 - ((N-2)/2)^2 K}. \quad (36)$$

The initial conditions corresponding to the diagonal form of the Hamiltonian at time $\eta = \eta_0$ if $p_{can,\lambda}^2(\eta_0) > 0$ and with normalization (14) can be written in the form

$$g'_\lambda(\eta_0) = ((N-2)c/2 + i p_{can,\lambda}(\eta_0)) g_\lambda(\eta_0), \quad |g_\lambda(\eta_0)| = 1/\sqrt{p_{can,\lambda}(\eta_0)}. \quad (37)$$

Since for the canonical Hamiltonian with $p_{can,\lambda}^2(\eta) > 0$ the equation (31) with substitution $p_\lambda \rightarrow p_{can,\lambda}$ is correct then functions $p_{can,\lambda}(\eta)$ mean the energies of quasiparticles corresponding to the canonical Hamiltonian diagonal form. One should note that $p_{can,\lambda}(\eta)$ energies differ from oscillator frequency $\Omega(\eta)$ similar to the case of the metrical Hamiltonian.

Let us consider the density problem of created in nonstationary metric pairs of quasiparticles in diagonalization procedure for the canonical Hamiltonian. For asymptotic solutions of equation (8) (see [14]), normalized according to (14), we can obtain from (19), (20) and (34), (35) that $E_J \sim \lambda$ and for nonstationary metric $|F_J(\eta)| \sim |Q(\eta)|$ in $\lambda \rightarrow \infty$. Therefore according to (29) that is corrected with substitution of $p_\lambda \rightarrow p_{can,\lambda}$, we have $|\beta_\lambda|^2 \sim \lambda^{-2}$. Consequently, the density of created quasiparticles, proportional to integral in (33), is infinite.

So, in diagonalization procedure both, for the metrical and the canonical Hamiltonian in nonconformal scalar field there is a problem of infinite density

of quasiparticles created in nonstationary metric. In both cases the energies of corresponding quasiparticles differ from the oscillator frequency of the wave equation. It will be shown below that these difficulties are absent in the case of the Hamiltonian corresponding to the modified energy-momentum tensor

$$T_{ik}^{mod} = T_{ik}^{can} - 2\xi_c [R_{ik} + \nabla_i \nabla_k - g_{ik} \nabla_j \nabla^j] \phi^* \phi. \quad (38)$$

From the definition (38) it is clear that for conformal scalar field (i.e. if $\xi = \xi_c$) T_{ik}^{mod} coincide with metrical energy-momentum tensor (5). The structure of the Hamiltonian constructed by T_{ik}^{mod} like (16) is

$$\begin{aligned} H^{mod}(\eta) = & \int d^{N-1}x \sqrt{\gamma} \left\{ \tilde{\phi}^{*'} \tilde{\phi}' + \gamma^{\alpha\beta} \partial_\alpha \tilde{\phi}^* \partial_\beta \tilde{\phi} + \right. \\ & \left. + [m^2 a^2 - \Delta \xi a^2 R + ((N-2)/2)^2 K] \tilde{\phi}^* \tilde{\phi} \right\}. \end{aligned} \quad (39)$$

It can be written in form (18), but in that case $Q(\eta) = 0$ and $D_\lambda(\eta) = \Omega^2(\eta)$. Under its diagonalization by $b_J^{*(\pm)}, b_J^{(\pm)}$, operators we have (31) with change $p_\lambda \rightarrow \Omega$. Oscillator frequency $\Omega(\eta)$ then coincide with the energy of corresponding particles. The initial conditions for $g_\lambda(\eta)$ corresponding to the diagonal form at time moment η_0 by operators $a_J^{*(\pm)}, a_J^{(\pm)}$ (17) are

$$g'_\lambda(\eta_0) = i \Omega(\eta_0) g_\lambda(\eta_0), \quad |g_\lambda(\eta_0)| = 1/\sqrt{\Omega(\eta_0)}. \quad (40)$$

They coincide with initial conditions used in [7] if $\arg g_\lambda(\eta_0) = 0$ is fixed. In radiation dominated background ($R = 0$) they coincides with conditions used in [6, 15].

Let us show finite density of the created in nonstationary metric particles corresponding to the diagonal form of Hamiltonian, built by T_{ik}^{mod} . It is necessary to find asymptotic in $\lambda \rightarrow \infty$ of the function $|\beta_\lambda(\eta)|^2$. The functions $\beta_\lambda(\eta)$ and $\alpha_\lambda(\eta)$, being the solutions (28) and satisfying identity (27), can be found in the form

$$\beta_\lambda(\eta) = \frac{i}{2} \frac{e^{i\Theta(\eta_0, \eta)}}{\sqrt{\Omega}} \left(g'(\eta) - i \Omega g(\eta) \right), \quad (41)$$

$$\alpha_\lambda(\eta) = \frac{i}{2} \frac{e^{i\Theta(\eta_0, \eta)}}{\sqrt{\Omega}} \left(g^{*'}(\eta) - i \Omega g^*(\eta) \right), \quad (42)$$

where $\Theta(\eta_1, \eta_2) = \int_{\eta_1}^{\eta_2} \Omega(\eta) d\eta$. In consequence of (41–42) and equation (8) the functions $s_\lambda(\eta) = |\beta_\lambda(\eta)|^2$ and $f_\lambda(\eta) = 2 \alpha_\lambda(\eta) \beta_\lambda(\eta) \exp[-2i \Theta(\eta_0, \eta)]$ satisfy

to the system of equations:

$$\begin{cases} s'_\lambda(\eta) = \frac{\Omega'}{2\Omega} \text{Re} f_\lambda(\eta), \\ f'_\lambda(\eta) + 2i\Omega f_\lambda(\eta) = \frac{\Omega'}{\Omega} (1 + 2s_\lambda(\eta)). \end{cases} \quad (43)$$

Taking into account the initial condition $s_\lambda(\eta_0) = f_\lambda(\eta_0) = 0$ (as $\beta_\lambda(\eta_0) = 0$) we write the system of differential equations (43) in the equivalent form of the system of Volterra integral equations

$$f_\lambda(\eta) = \int_{\eta_0}^{\eta} w(\eta_1) (1 + 2s_\lambda(\eta_1)) \exp[-2i\Theta(\eta_1, \eta)] d\eta_1, \quad (44)$$

$$s_\lambda(\eta) = \frac{1}{2} \int_{\eta_0}^{\eta} d\eta_1 w(\eta_1) \int_{\eta_0}^{\eta_1} d\eta_2 w(\eta_2) (1 + 2s_\lambda(\eta_2)) \cos[2\Theta(\eta_2, \eta_1)], \quad (45)$$

where $w(\eta) = \Omega'(\eta)/\Omega(\eta)$. To find asymptotic $s_\lambda(\eta)$ let us consider the first iteration in equation (45) and as $\Theta(\eta_2, \eta_1) \rightarrow \lambda(\eta_1 - \eta_2)$ in $\lambda \rightarrow \infty$ we will have

$$s_\lambda(\eta) \approx \frac{1}{4} \left| \int_{\eta_0}^{\eta} w(\eta_1) \exp(2i\lambda\eta_1) d\eta_1 \right|^2. \quad (46)$$

Therefore $s_\lambda \sim \lambda^{-6}$ and integral in (33) is finite.

Thus in this case the density of created particles is finite for 4 dimensional space-time. If $t \ll 1/m$ in special case the density of created nonconformal particle was calculated in [7].

5. The Modified Hamiltonian as Canonical

We'll show that the modified Hamiltonian (39) can be obtained in homogeneous isotropic space as canonical under the certain choice of variables describing scalar field. Let's add N -divergence $\frac{\partial J^i}{\partial x^i}$, to the lagrangian density (3), where in (η, \mathbf{x}) system of coordinates N -vector $J^i = (\sqrt{\gamma} c \tilde{\phi}^* \tilde{\phi} (N-2)/2, 0, \dots, 0)$. The movement equations (1) are invariant under this addition. Let us choose $\tilde{\phi}(x) = a^{(N-2)/2}(\eta)\phi(x)$ and $\tilde{\phi}^*(x)$ as the field's variables. In these variables the equation (1) have a form (7). From the Lagrangian density $L^\Delta(x) = L(x) + \frac{\partial J^i}{\partial x^i}$ it is follows that the Hamiltonian

density $h(x) = \tilde{\phi}' (\partial L^\Delta)/(\partial \tilde{\phi}') + \tilde{\phi}^{*'} (\partial L^\Delta)/(\partial \tilde{\phi}^{*'}) - L^\Delta(x)$ takes the value

$$h(x) = \sqrt{\gamma} \left\{ \tilde{\phi}^{*'} \tilde{\phi}' + \gamma^{\alpha\beta} \partial_\alpha \tilde{\phi}^* \partial_\beta \tilde{\phi} + \right. \\ \left. + [m^2 a^2 - \Delta \xi a^2 R + ((N-2)/2)^2 K] \tilde{\phi}^* \tilde{\phi} \right\}. \quad (47)$$

That's why the Hamiltonian (39) is a canonical one for scalar field if $\tilde{\phi}(x)$ and $\tilde{\phi}^*(x)$ are chosen as the field's variables.

In the presented work the metrical, canonical and introduced modified Hamiltonian were investigated. It was shown that the density of particles created in nonstationary homogeneous isotropic space metric is finite only in the case of modified Hamiltonian (39) and the energies of such particles are equal to the oscillator frequency.

Modified energy-momentum tensor (38) introduced above coincide with the metrical one for conformal scalar field. In homogeneous isotropic space T_{ik}^{mod} results in modified Hamiltonian (39) that can be obtained as well as canonical under the special choice of field's variables.

One can see that considering a line combination of metrical (5) and canonical (4) tensors we can certainly get modified tensor (38) if the quasiparticles' energy coincides with oscillator frequency. It should be stressed that metrical energy-momentum tensor can't be changed to T_{ik}^{mod} in the right-hand sides of Einstein's equations because T_{ik}^{mod} is not covariant conservation. But under corpuscular interpretation of nonconformal scalar field and when diagonalization procedure is used the Hamiltonian (39) constructed by T_{ik}^{mod} is preferable in comparison with the metrical Hamiltonian.

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